Quantum stochastic differential equation is unitary equivalent to a symmetric boundary value problem in Fock space *

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Abstract

We show a new remarkable connection between the symmetric form of a quantum stochastic differential equation (QSDE) and the strong resolvent limit of Schrödinger equations in Fock space: the strong resolvent limit is unitary equivalent to QSDE in the adapted (or Ito) form, and the weak limit is unitary equivalent to the symmetric (or Stratonovich) form of QSDE.

We prove that QSDE is unitary equivalent to a symmetric boundary value problem for the Schrödinger equation in Fock space. The boundary condition describes standard jumps of the phase and amplitude of components of Fock vectors belonging to the range of the resolvent. The corresponding Markov evolution equation (the Lindblad or Markov master equation) is derived from the boundary value problem for the Schrödinger equation.

1 Introduction

The last two decades show a valuable progress in quantum probability theory and applications [1]–[5]. It was discovered that fundamental constructions of classical probability theory, such as central limit theorems, conditional expectations, martingales, stopping times, the Markov property, and Markov evolution equations,

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have noncommutative generalizations [5]–[6]. The quantum stochastic differential equation (QSDE) is a noncommutative generalization of the Ito stochastic equation suitable for describing irreversible Markov evolution in operator algebras [7]–[11]. The QSDE and the Schrödinger equation describe a unitary evolution of a quantum system and its environment. The solution of the Schrödinger equation is a one-parameter unitary group U_t, and the solution of the QSDE is a unitary cocycle $u(s,t)$, i.e. an interval-dependent family of unitary operators with one of two composition laws $u(s,\tau)u(\tau,t) = u(s,t),\ s \le \tau \le t$ for the right cocycle, or $u(t,\tau)u(\tau,s) = u(t,s)$ for the left cocycle. From a mathematical viewpoint, this structure difference between U_t and $u(s,t)$ is superficial and physically unobservable, because $u(s,t)$ is usually an interaction representation of some unitary group U_{t-s}.

A deep distinction between a QSDE and a typical Schrödinger equation is that the QSDE necessarily involves a singular component in its formal Hamiltonian, i.e. it can be regarded, up to unitary equivalence, as a Schrödinger equation with Hamiltonian operator perturbed by singular bilinear forms (see [12]–[14]).

Such Hamiltonians in $\mathcal{H} \otimes \Gamma^S(L_2(I\!R))$ appear as strong resolvent limits ($r - \lim$ or $srs - \lim$) of self-adjoint operators $\widehat{H}_\alpha = H \otimes I + I \otimes \widehat{\mathbf{E}} + H_{int}^{(\alpha)}$ depending on a scaling parameter $\alpha \in (0,1]$ such that

$$H_{int}^{(\alpha)} = R^* \otimes A(f_\alpha) + R \otimes A^+(f_\alpha) + K \otimes A^+(g_\alpha)A(g_\alpha),$$

where $K = K^* > 0,\ (R^*)^* = R,\ f, g \in L_2(I\!R),\ f_\alpha(\omega) = f(\alpha\omega),\ g_\alpha(\omega) = g(\alpha\omega)$, $f(0) = g(0) = (2\pi)^{-1/2}$. The family of quadratic forms $H_{\alpha,*}[h \otimes \psi(v)] = (h \otimes \psi(v), \widehat{H}_\alpha\, h \otimes \psi(v))$ converges clearly to the quadratic form

$$\begin{aligned} H_*[h \otimes \psi(v)] &= e^{||v||^2}\Big((h, Hh) + ||h||^2(\psi(v), \widehat{\mathbf{E}}\psi(v)) \\ &+ (Rh, h)\tilde{v}(0) + (h, Rh)\overline{\tilde{v}(0)} + (h, Hh)|\tilde{v}(0)|^2\Big), \end{aligned}$$

with the singular component vanishing on the total subset consisting of vectors $h \otimes \psi(v) \in \mathcal{H} \otimes \Gamma^S(L_2(I\!R))$ such that $v \in C_0^\infty(I\!R),\ \tilde{v}(0) = 0,\ h \in \mathrm{dom}\, K \cap \mathrm{dom}\, R \cap \mathrm{dom}\, R^* \cap \mathrm{dom}\, H$.

We prove, for commuting coefficients $K,\ R$, and H that, in the strong resolvent sense, $\widehat{H}_\alpha$ converges to a quadratic form with another regular and singular components:

$$\begin{aligned} \mathbf{H}_*[h \otimes \psi(v)] &= e^{||v||^2}\Big((h, H_0 h) + ||h||^2(\psi(v), \widehat{\mathbf{E}}\psi(v)) \\ &+ (h, H_1 h)\tilde{v}(0) + (h, H_2 h)\overline{\tilde{v}(0)} + (h, H_3 h)|\tilde{v}(0)|^2\Big), \end{aligned}$$

where

$$H_0 = H - R^*\frac{K}{4+K^2}R + R^*\frac{2i}{4+K^2}R \stackrel{\text{def}}{=} -iG, \quad H_1 = R^*\frac{2}{2-iK} \stackrel{\text{def}}{=} iL^*W,$$

$$H_2 \stackrel{\text{def}}{=} \frac{2}{2-iK}R = -iL, \quad H_3 = \frac{2K}{2-iK} = i(I-W), \quad W = \frac{2+iK}{2-iK}. \qquad (1.1)$$

Ill-defined operators A, A^+, and Λ that correspond formally to singular quadratic forms

$$A_*[\psi(v)] = e^{||v||^2}\widetilde{v}(0), \quad A_*^+[\psi(v)] = e^{||v||^2}\overline{\widetilde{v}(0)}, \quad \Lambda_*[\psi(v)] = e^{||v||^2}|\widetilde{v}(0)|^2$$

generate well-defined operator-valued measures

$$M_1(T) = A(T) = \int_T dt\, J_t^* A J_t, \quad M_2(T) = A^+(T) = \int_T dt\, J_t^* A^+ J_t,$$

$$M_3(T) = \Lambda(T) = \int_T dt\, J_t^* \Lambda J_t$$

which, up to unitary equivalence, act as fundamental *creation, annihilation and number* processes in the Hudson–Parthasaraty framework:

$$A_*(T)[\psi(v)] = e^{||v||^2}\int_T dt\, \widetilde{v}(t), \quad A_*^+(T)[\psi(v)] = e^{||v||^2}\int_T dt\, \overline{\widetilde{v}(t)},$$

$$\Lambda_*(T)[\psi(v)] = e^{||v||^2}\int_T dt\, |\widetilde{v}(t)|^2,$$

where $\widetilde{v}(t) = \mathcal{F}_{\omega\to t}v(\omega)$ is the Fourier transform of $v(\omega)$. Set $M_0(T) = \operatorname{mes} T$ and denote by $M(T) = \sum_0^3 L_k \otimes M_k(T)$ an operator-valued measure in the QSDE

$$d\,u(0,t) = u(0,t)M(dt_+), \;\; M(dt_+) = M(t, t+dt).$$

One of our main results explains a connection between the unitary group U_t and the solution $u(s,t)$ of QSDE:

$$U_t = e^{it\widehat{\mathbf{H}}} = s - \lim_{\alpha\to 0} e^{it\widehat{H}_\alpha} = u(0,t)J_t, \qquad (1.2)$$

$$M(T) = \sum_0^3 L_k \otimes M_k(T) = i\int_T d\tau (J_\tau \widehat{\mathbf{H}} J_\tau^* - I\otimes\widehat{\mathbf{E}}) \qquad \forall T \in \mathcal{B}(I\!R),$$

where $L_k = iH_k$ (see equation (1.1)), and $J_t = I\otimes e^{it\widehat{\mathbf{E}}}$, $\widehat{\mathbf{E}} = \int d\omega\, \omega\, a^+(\omega)\, a(\omega)$ is an environment energy operator generating an interaction representation of the unitary evolution

$$u(s,t) = J_s U_{t-s} J_t^*.$$

A remarkable observation is that the weak limit $\widehat{H} = w - \lim \widehat{H}_\alpha \neq \widehat{\mathbf{H}} = r - \lim \widehat{H}_\alpha$ can be derived by a symmetrization of the QSDE: $u(0,t)M(dt_+) = u(0,t)N(dt)$, where $M(dt_+) = M(t, t+dt)$ is the stochastic differential of the adapted equation, and $N(dt) = N(t-dt/2, t+dt/2)$ is the stochastic differential of the corresponding symmetric equation. A relation between the symmetric differential N and the corresponding adapted differential M, proved in §2, has the form of an integral equation

$$N(T) + \frac{1}{2}\int_T M(dt_+)N(dt_+) = M(T) \qquad \forall T \in \mathcal{B}(I\!\!R), \tag{1.3}$$

and an unexpected fact is that the weak limit $\widehat{H}$ contributes to the measure $N(T)$ of the symmetric differential:

$$N(T) = i\int_T d\tau (J_\tau \widehat{H} J_\tau^* - I \otimes \hat{\mathbf{E}}) \quad \text{or} \quad \widehat{H} = I \otimes \hat{\mathbf{E}} + \frac{1}{i \operatorname{mes} T}\int_T J_\tau^* N(d\tau) J_\tau.$$

Equation (1.3) implies the Hudson–Parthasaraty necessary condition for a solution of QSDE to be unitary [2]:

$$L_0 = iH_s - \frac{1}{2}L^*L, \quad L_1 = -L^*W, \quad L_2 = L, \quad L_3 = W - I$$

where L and W are related to operators K and R as in (1.1), and $H_s = H - i/2L^*(I-W)(I+W)^{-1}L = H - R^*K(4+K^2)^{-1}R$. We would like to remark that the solution of (1.3) differs from solutions of the equation $M(dt) = \exp\{N(dt)\} - I$ derived in [15] for the symmetric operator-valued measure $iN(T)$.

Very technical assumptions sufficient for the unitary property of solutions of QSDE were obtained in the last decade basically by perturbation methods [7], [15]–[18], but symmetric operators responsible for the unitary property of solutions of QSDE were not discovered. The difficulties are connected with the violation of the group property by solutions of QSDE (see [21]), and with the violation of the symmetric property of the form-generators considered on the domain of Fock vectors with smooth components. More precisely (see [20]), the formal generator of the Schrödinger equation, which is unitary equivalent to QSDE, reads as a dissipative operator, perturbed by nonsymmetric singular (in the sense of [12]–[14]) bilinear form.

In this paper we consider a class of QSDE which appears as an interaction representation of the strong resolvent limit of the Schrödinger equations $\partial_t \psi_t = i\widehat{H}_\alpha \psi_t$ in the Fock space parameterized by the scaling variable α. The basic property of the limit resolvent is the existence of standard *jumps of amplitude and phase* of Fock vector components belonging to $\mathcal{R}$:

$$(\widehat{N}+1)^{-1}(I \otimes A(\delta_+) - W \otimes A(\delta_-))\Psi = (L \otimes I)\,\Psi, \quad \forall \Psi \in \mathcal{R} \tag{1.4}$$

where $W = (2+iK)/(2-iK)$, $L = 2i/(2-iK)\,R$, $\Psi = \{\Psi_0, \ldots, \Psi_n(\omega_1, \ldots, \omega_n), \ldots\}$,

$$A(\delta_\pm)\widetilde{\Psi}_n(\tau) = \lim_{\varepsilon \to \pm 0} \sum_{k=1}^{n} \widetilde{\Psi}_{n+1}(\tau_1, \ldots, \tau_{k-1}, \epsilon, \tau_k, \ldots, \tau_n), \quad \widehat{N}\Psi_n(\omega) = n\Psi_n(\omega),$$

$\widetilde{\Psi}_n(\tau_1, \ldots, \tau_n)$ is the Fourier transform $\mathcal{F}_{\omega\to\tau}$ of n-th component of Fock vector Ψ.

The support of discontinuities of vectors from R coincides with the support of singularities of the quadratic form associated to the formal generator of the limit unitary group. Our main observation is that the generator of group derived on $\mathcal{R}$ (i.e. in the set of Fock vectors with the standard discontinuity (1.4))

$$\widehat{\mathbf{H}} = H_0 \otimes I + I \otimes \hat{\mathbf{E}} + iL^* W \otimes A(\delta_-), \quad \hat{\mathbf{E}} = \mathcal{F}_{\tau\to\omega} \int_{I\!R\setminus\{0\}} d\tau a^+(\tau) a(\tau)\, i\partial_\tau \mathcal{F}^{-1}_{\omega\to\tau}, \tag{1.5}$$

is a symmetric operator, that is $(\Phi, \widehat{\mathbf{H}}\Psi) = (\widehat{\mathbf{H}}\Phi, \Psi)$ $\forall \Psi, \Phi \in \mathcal{R}$; its symmetric property can be verified independently under assumptions less restrictive than used for an explicit construction of the resolvent.

The algebraic computations permit to derive the Markov master equation directly from the boundary problem for the Schrödinger equation.

The structure of the paper is the following. First, in §2 an algebraic analog of the Markov property is introduced for operator–valued processes in the Fock space. The processes with this property are called *interval adapted*. It is proved prove that the assumption (1.3) is sufficient to transform an adapted QSDE to the symmetric form. In §3 we construct an explicit solution of the QSDE with commuting coefficients and prove (1.1)–(1.2) for this particular case. A class of explicitly solvable Schrödinger equations in the Fock space is analysed in §4. A direct computation shows that QSDE in the weak Hudson–Parthasaraty form coincides with the interaction representation of the limit Schrödinger equation. Next, in §4 we consider the properties of the resolvent of the limit Schrödinger equation, and prove that it ranges in a set of Fock vectors with components that satisfy a standard discontinuity conditions (1.4). In §5 the Markov master equation is derived from the boundary value problem for the limit Schrödinger equation. The main results of this paper were included in [20] and [25].

This work, in its final stage, was inspired by intensive discussions with Prof. L. Accardi during a short stay of the author in V. Volterra Center in February 1996; our interest in the problem of the parameterization of generators of QSDE by self-adjoint operators was initiated long ago by the papers [1], [10], [4]-[19].

2 Interval–adapted processes in Fock space

Let $\mathcal{H}$ and $\mathcal{H}_E$ be Hilbert spaces. The Hilbert space of a quantum system and its environment has a form of the tensor product $I\!H = \mathcal{H} \otimes \Gamma^S$, where $\mathcal{H}$ describes states of the quantum system, and $\Gamma^S = \Gamma^S(\mathcal{H}_E)$ is the symmetric Fock space describing a state of the environment. We denote by $(\cdot, \cdot)$, $||\cdot||$ a scalar product and a norm in the corresponding Hilbert space. The scalar product in Γ^S is generated by the scalar product in $\mathcal{H}_E$:

$$(F, F') = \overline{F}_0\, F' + \sum \frac{1}{n!}(F_n, F'_n), \quad F = \{F_0, F_1, \ldots\} \in \Gamma^S, \quad F_0 \in \mathbb{C}, \quad F_n \in \otimes_1^n \mathcal{H}_E.$$

Let $\mathcal{E} \subset \Gamma^S$ be a total subset of coherent (exponential) vectors $\psi(f) = \{1,\, f,\, f \otimes f,\, \ldots, f \otimes \ldots \otimes f,\, \ldots\}$, $f \in \mathcal{H}_E$ (see [6], [5]). A symmetric tensor product is defined on the total subset $\mathcal{P}$ of polinomial vectors $F_j = f^{\otimes j}$ and $F'_{n-j} = f'^{\otimes n-j}$ by the equation

$$F \otimes F' = \Big\{ F_0 \cdot F'_0,\;\; F_0 \otimes F'_1 + F'_0 \otimes F_1,\;\; \ldots, \sum_{j=0}^{n} \mathbf{S}_n(F_j \otimes F'_{n-j}),\; \ldots \Big\}, \tag{2.1}$$

where $\mathbf{S}_n$ is a sum over all permutations of arguments in multipliers $f,\, f'$. This definition can be extended by continuity to the linear span of $\mathcal{P}$ and to $\overline{\mathrm{Span}\,\mathcal{P}} = \Gamma^S$.

Definition (2.1) implies the relation $\psi(f) \otimes \psi(v) = \psi(f + v)$ for coherent vectors. Hence, $\psi(v) = \otimes_i \psi(\widehat{\pi}_{T_i} v)$ for any (finite) identity decomposition or complete system of projectors $\{\widehat{\pi}_{T_j}\}_{T_j \in \mathcal{T}} \subset \mathcal{B}(\mathcal{H}_E)$, parameterized by subsets of a measurable space $(\mathcal{T}, \mathbf{T})$:

$$\widehat{\pi}^*_T = \widehat{\pi}_T = \widehat{\pi}^2_T, \quad \forall T \in \mathcal{T}, \quad \widehat{\pi}_{T_1}\widehat{\pi}_{T_2} = \widehat{\pi}_{T_2}\widehat{\pi}_{T_1} = 0, \quad \forall T_1, T_2 :\, T_1 \cap T_2 = \emptyset.$$

The Fock space Γ^S has a property of the tensor decomposition for any finite complete system of projectors. In particular, $\Gamma^S(\mathcal{L}_2(I\!R)) = \otimes_j \Gamma^S_{T_j}$, $\quad \Gamma^S_{T_j} = \Gamma^S(\mathcal{F}\mathcal{L}_2(T_j))$ for any finite disjoint partition $\{T_j\}$, and $\pi_T = \mathcal{F}^*\, I_T\, \mathcal{F}$ for any unitary operator $\mathcal{F}$, where $\mathcal{T} = I\!R$, $\mathcal{H}_E = \mathcal{L}_2(I\!R)$; I_T stands for an multiplication operator by the indicator function of a measurable subset T.

Later $I\!H^\pi_T$ denotes a product $\mathcal{H} \otimes \Gamma^S$ equipped by the system of orthogonal projectors $\{\widehat{\pi}_T\}_{T \in \mathcal{T}}$, and $I\!H_T$ is the subspace $\mathcal{H} \otimes \Gamma^S_T$, where $\Gamma^S_T = \Gamma^S(\pi_T \mathcal{H}_E)$. The operator-valued family $u(\cdot) : \mathcal{T} \to \mathcal{B}(I\!H)$ is called $(\widehat{\pi}, \mathcal{T})$-*adapted* if for any $T \in \mathcal{T}$ the following two assumptions hold:

$$u(T)h \otimes \psi(\widehat{\pi}_T v) \in \mathcal{H} \otimes \Gamma^S_T, \quad u(T)h \otimes \psi(v) = \psi(\widehat{\pi}_{T^a} v) \otimes \{u(T)h \otimes \psi(\widehat{\pi}_T v)\},$$

where $T^a = I\!R \backslash T$; that is the operators $u(T)$ leave invariant vectors from $I\!H_{T^a} \in I\!H^\pi_T$ (and the subset $I\!H_T = \mathcal{H} \otimes \Gamma^S_T$). In what follows the Fourier transform is used as

a unitary operator $\mathcal{F}$, and the corresponding representation of multiplication by indicator functions play the role of projectors. In that case we call the family of operators $\{u(T)\}$, $T \in \mathcal{B}(I\!\!R)$ *interval adapted* (see [16]):

$$(\varphi \otimes \psi(f),\, u(T)\, g \otimes \psi(v)) = \exp\{(f, \widehat{\pi}_{T^a} v)\}(\varphi \otimes \psi(\widehat{\pi}_T\, f), u(T)\, g \otimes \psi(\widehat{\pi}_T\, v)), \quad (2.2)$$

with $\varphi, g \in \mathcal{H}$, $\psi(f), \psi(v) \in \mathcal{E}$, $T^a = I\!\!R \setminus T$. Definition (2.2) means that the operator $u(T)$ changes only the part of exponential vector from Γ^S_T. Denote by $\mathcal{B}(I\!\!H_T) = \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma^S_T)$ an algebra of all bounded operators acting as in (2.2). Equation (2.2) can be verified in the weak topology, and hence it holds for the adjoint operator $u^*(T)$ as a hereditary property of $u(T)$. Relation (2.2) implies the commutativity of the components from $\Gamma^S_{T^a}$ of a coherent vector and interval adapted operator families $u(T)$, $M(T)$:

$$(\varphi \otimes \psi(f), u(T_1)\, M(T_2)\, g \otimes \psi(v)) = \exp\,(f, \widehat{\pi}_{T^a} v)$$

$$\times\Big(\{u^*(T_1)\, \varphi \otimes \psi(\widehat{\pi}_{T_1}\, f)\} \otimes \psi(\widehat{\pi}_{T_2}\, f), \{M(T_2) g \otimes \psi(\widehat{\pi}_{T_2}\, v)\} \otimes \psi(\widehat{\pi}_{T_1}\, v)\Big), \quad (2.3)$$

where $T_a = I\!\!R \setminus \{\cup T_i\}$, $T_1 \cap T_2 = \emptyset$. An important example of interval adapted operators is *creation, annihilation* and *conservation* processes in the Fock space:

$$(\psi(f), A(T)\psi(v)) = e^{(f,v)}(\widetilde{I}_T, v), \quad (\psi(f), A^+(T)\psi(v)) = e^{(f,v)}(f, \widetilde{I}_T).$$

$$(\psi(f), \Lambda(T)\psi(v)) = (f, \widehat{\pi}_T v)\exp\{(f, v)\}, \quad (2.4)$$

with $\widehat{\pi}_T = \mathcal{F}^*_{t\to\omega} I_T(t) \mathcal{F}_{\omega\to t}$.

Let $a^+(\omega)$, $a(\omega)$, $\omega \in I\!\!R$ be creation and annihilation operator densities, $\widehat{\mathbf{E}} = \int \omega\, a^+(\omega)\, a(\omega)\, d\omega \in \mathcal{C}(\Gamma^S)$ ($\mathcal{C}(H)$ denotes a set of *closed* operators in a Hilbert space H) be a generator of the Journé unitary transform: $J_t = \exp\{it\widehat{\mathbf{E}}\}$. The group of unitary operators $J_t \psi(v) = \psi(e^{i\omega t} v)$ is uniquely defined, and from the canonical commutation relations we have:

$$a^+_t(\omega) = J^*_t\, a^+(\omega)\, J_t = e^{-i\omega t} a^+(\omega), \quad J_t\, a^+(\omega)\, J^*_t = e^{i\omega t} a^+(\omega). \quad (2.5)$$

Let $\widetilde{W}^p_2(I\!\!R)$ be the Sobolev space with the scalar product $\langle f, v\rangle = (f(\omega), v(\omega)(1 + \omega^2)^{p/2})$, $p \in I\!\!R$. Denote by $A = A(1) = \int_{I\!\!R} a(\omega)\, d\omega$, $A^+ = A^+(1) = \int_{I\!\!R} \overline{a^+(\omega)}\, d\omega$, and $\Lambda = A^+ A$, quadratic forms in $\widetilde{W}^2_1(I\!\!R)$, so that $(\psi(f), A^+ A\psi(v)) = 2\pi\, \overline{\widetilde{f}(0)}\widetilde{v}(0) e^{(f,v)}$,

$$(\psi(f), A\psi(v)) = \sqrt{2\pi}\, \widetilde{v}(0) e^{(f,v)}, \qquad (\psi(f), A^+\psi(v)) = \sqrt{2\pi}\, \overline{\widetilde{f}(0)} e^{(f,v)}.$$

The quadratic forms A, A^+, Λ are not closable in $\Gamma^S(\mathcal{L}_2(I\!\!R))$ and vanish on the dense set generated by coherent vectors $\{\psi(v) : \widetilde{v}(0) = 0\}$. They are closable in

$\Gamma^S(\widetilde{W}_2^1(I\!R))$ ([12, 13]) and define closed operators from $\Gamma^S(\widetilde{W}_2^1(I\!R))$ to $\Gamma^S(\widetilde{W}_2^{-1}(I\!R))$. Operators (2.4) can be expressed as operators, corresponding to densely defined closable quadratic forms:

$$A(T) = \frac{1}{\sqrt{2\pi}} \int_T A_t\, dt, \quad A^+(T) = \frac{1}{\sqrt{2\pi}} \int_T A_t^+\, dt, \quad \Lambda(T) = \frac{1}{2\pi} \int_T A_t^+ A_t\, dt \quad (2.6)$$

with $A_t = J_t^*\, A\, J_t,\ A_t^+ = J_t^*\, A^+\, J_t$ and $\Lambda_t = J_t^*\, \Lambda\, J_t$. In the standard notation, the operator $A(T)$ coincides with the annihilation operator $A(\widetilde{I}_T)$, where $\widetilde{I}_T(\omega)$ is the Fourier transform of the indicator function of a bounded set $T \in \mathcal{B}(I\!R)$.

Let $M(T) = \sum L_j \otimes M_j(T)$ be an additive function so that

$$M_0(T) = I \cdot \mathrm{mes}\, T, \quad M_1(T) = A(T), \quad M_2(T) = A^+(T), \quad M_3(T) = \Lambda(T), \quad (2.7)$$

$L_j \in \mathcal{C}(\mathcal{H}),\ M_j(T) \in \mathcal{C}(\Gamma^S)$. Assume there exist a joint dense domain $D_0 \in \mathcal{H}$ for L_j and their products. Let $g \in D_0,\ h \in \mathcal{H},\ |\widetilde{f}|, |\widetilde{v}| \leq 1$. Identities (2.4) generate the equation for correlators

$$(h \otimes \psi(f), M(T)\, g \otimes \psi(v)) = e^{(f,v)} \cdot \sum_{i=0}^{3} (h, L_i g) \cdot ((\widetilde{f})^{\alpha_i}, I_T\, \widetilde{v}^{\beta_i}) = O(\mathrm{mes}\, T) \quad (2.8)$$

with $\alpha_i = 1$ for $i = 2, 3$ and $\beta_i = 1$ for $i = 1, 3$; $\alpha_i = \beta_i = 0$ in other cases. It follows from here that the family $M(T)$ is interval adapted in the sense of definition (2.2), and that the mean value is absolutely continuous with respect to the standard Lebesgue measure:

$$\frac{d}{dt}(h \otimes \psi(f), M(0,t)\, g \otimes \psi(v)) = e^{(f,v)} \cdot \sum_{i=0}^{3} (h, L_i g) \overline{(\widetilde{f}(t))^{\alpha_i}}\, \widetilde{v}(t)^{\beta_i} \in L_1(I\!R). \quad (2.9)$$

These observations are important for a rigorous definition of the QSDE in the weak form [2].

Consider the operator-valued stochastic differential equation

$$du(\tau, t) = u(\tau, t)\, M(dt_+), \quad \tau \leq t, \quad s - \lim_{T \to \emptyset} u(T) = I \quad (2.10)$$

(see [2]), where $M(T)$ is an additive function on $\mathcal{B}(I\!R)$ which ranges in the set of interval adapted closable operators wth a common dense domain $\mathcal{D} \subseteq I\!H$. The strongly continuous interval adapted cocycle $u(\cdot) : \mathcal{B}(I\!R) \to \mathcal{B}(I\!H)$ is called *a weak solution* of equation (2.10) if for any $\varphi \in \mathcal{D}$ there exist a limit

$$\int_T u(s, \tau) M(d\tau_+)\varphi = \omega - \lim_{N \to \infty} \sum_{j \in J_N(T)} u(s, t_j) M(t_j, t_{j+1})\varphi,$$

so that $u(s,t)\varphi = \varphi + \int_s^t u(s,\tau)M(d\tau_+)\varphi$, with $t_j = j2^{-N},\ J_N(T) = \{j:\ t_j \in T\}$. From the bound (2.9) follows an existence of time derivatives of the cocycle u

$$\frac{d}{dt}(h\otimes\psi(f), u(\tau,t)g\otimes\psi(v)) = \sum_i \Big(h\otimes\psi(f), u(\tau,t)L_i g\otimes\psi(v)\Big)\overline{\widetilde{f}(t)^{\alpha_i}}\widetilde{v}(t)^{\beta_i} \quad (2.11)$$

$h \in \mathcal{H},\ g \in D_0,\ f, v \in \widetilde{W}_2^1(I\!R)$. Equation (2.11) is called *a weak* form of QSDE (2.10). In this section we aim to derive a symmetric form of the equation (2.10), that is to construct an operator-valued measure $N(T)$ such that

$$w - \lim \varepsilon^{-1}\, u(t)\, M(t, t+\varepsilon)\varphi = w - \lim \varepsilon^{-1}\, u(t)\, N(t-\varepsilon/2, t+\varepsilon/2)\varphi \quad \forall\varphi \in \mathcal{D}.$$

We start with a suitable choice of operator-valued measures $M(\cdot)$ in equation (2.10). Let $\mathcal{D}$ be a total subset in $I\!H = I\!H_{\mathcal{T}}^{\pi}$, $\mathcal{T} = \mathcal{B}(I\!R)$, $\widehat{\pi}_T = \mathcal{F}^*_{\tau\to\omega} I_T(\tau)\mathcal{F}_{\omega\to\tau}$. The main property of the projector $\widehat{\pi}_T$ follows from the properties of the Fourier transform:

$$e^{i\omega t}\widehat{\pi}_T e^{-i\omega t} = \widehat{\pi}_{T+t}, \quad \mathcal{F}_{\omega\to t} v = (2\pi)^{-1/2}\int_{I\!R} d\omega\, e^{-i\omega t} v(\omega), \quad T \in \mathcal{B}(I\!R), \quad t,\omega \in I\!R.$$

Example 2.1. For equation (2.10) with bounded coefficients $\{L_j\}$ we put $\mathcal{D} = \mathcal{H}\otimes\mathcal{E}_1$, where $\mathcal{E}_1 \subset \Gamma^S$ is a subset of coherent vectors $\psi(f)$ such that $|\widetilde{f}| \le 1$, and $f \in \mathcal{L}_2(I\!R)$ is a finitely supported function. $\mathcal{E}_1$ is total in Γ^S ([6]) and $\mathcal{D}$ is total in $I\!H$. It is convenient to put $\mathcal{D} = D_0 \otimes \mathcal{E}_1$, for equations with unbounded coefficients, if there exist where is a dense joint domain $D_0 \subseteq \mathcal{H}$ for all L_k and their products.

Definition 2.1 *The vector subspace of additive interval adapted functions* $M : \mathcal{B}(I\!R) \to \mathcal{B}(I\!H_{\mathcal{T}}^{\pi})$ *is labelled by* $\mathcal{A}_{\mathcal{D}}$ *if for any* $\varphi \in \mathcal{D}$

(i) *there exist an upper bound*

$$\sup_{\{T_j\},\{\psi_j\}} \sum_j |(\psi_j, M(T_j)\varphi)| = \mu_T(\varphi), \quad (2.12)$$

where T *is a bounded Borelean subset of* $I\!R$, $\{T_j\}$ *is a disjoint partition of* T, $\{\psi_j \in I\!H_{T_j^a}\}$ *is a uniformly bounded family of vectors from* $I\!H$, $\sup_j ||\psi_j|| \le 1$, *and* μ_T *is a finite measure such that* $\mu_T = \mu_T(\varphi) = 0$ *if* $\mathrm{mes}\, T = 0$;

(ii) *there exist a derivative*

$$f_t(\psi,\varphi) = \frac{(\psi, M(dt_+)\varphi)}{dt} = \lim_{\varepsilon\to 0}\varepsilon^{-1}\,(\psi, M(t,t+\varepsilon)\,\varphi), \quad (2.13)$$

which is a continuous function from $I\!H \to \mathcal{L}_1^{loc}(I\!R)$ *for any fixed* $\varphi \in \mathcal{D}$.

Assumption (i) requires an existence of the weak absolute upper bound for integral sums for every natural N and partitions $\{T_j\}$; condition (ii) assumes an existence of the majorizing function $(\psi, M(T)\varphi)$ which is absolutely continuous w.r.t. the standard Lebesgue measure.

To fix a notation set $\widehat{\pi}_T = \mathcal{F}_{t\to\omega} I_T \mathcal{F}^*_{\omega\to t}$ as a system of projectors, and $\mathcal{B}(I\!R)$ as a Borelean σ-algebra $\mathcal{T}$. Denote $\delta_N = 2^{-N} = \tau_1$, $\tau_j = \tau_j^{(N)} = j\,2^{-N}$, $\quad J_N(T) = \{j : \tau_j^{(N)} \in T\}$.

Lemma 2.1 *For every $L \in \mathcal{A}_{\mathcal{D}}$, $\varphi \in \mathcal{D}$ and the strongly continuous interval adapted bounded cocycle $u(t) = u(0,t)$ there exist an interval adapted weak limits*

$$\lim_{N\to\infty} \sum_{j\in J_N(T)} u(\tau_j^{(N)})\, L(\tau_j^{(N)}, \tau_{j+1}^{(N)})\varphi = \int_T u(\tau)\, L(d\tau_+)\varphi, \tag{2.14}$$

$$\lim_{N\to\infty} 2 \sum_{j\in J_N(T)} u(\tau_j^{(N)})\, L(\tau_j^{(N)}, \tau_j^{(N)} + \delta_{N+1})\varphi = \int_T u(\tau)\, L(d\tau_+)\varphi, \tag{2.15}$$

called adapted stochastic integrals.

Proof. We can apply the Lebesgue theorem on the dominated convergence to prove the equation (2.14) because there exist a uniform absolute bound (2.12) for sums:

$$J_N = \sum_{j\in J_N(T)} |(\psi, u(\tau_j)\, L(\tau_j, \tau_{j+1})\,\varphi)| \le \sup_N \sum_{j\in J_N(T)} |(\psi_j, L(\tau_j, \tau_{j+1})\,\varphi)| \le \mu_T(\varphi),$$

where $\psi_j = u^*(0,\tau_j)\,\psi$, and the limit $f_\tau(\psi, u^*(\tau)\,\varphi) = \lim \varepsilon^{-1}\,(u^*(\tau)\,\psi, L(\tau, \tau+\varepsilon)\,\varphi)$ belongs to L_1^{loc} because of (2.13). Hence, $\lim J_N = J_T(\psi,\varphi) = \int_T d\tau\, f_\tau(u^*(\tau)\,\psi, \varphi)$. Since $f_\tau(u^*(\tau)\,\cdot, \varphi) : I\!H \to L_1^{loc}(I\!R)$ is a continuous function for any $\varphi \in \mathcal{D}$, there exist a unique element $\varphi_T \in I\!H$ so that $(\psi, \varphi_T) = J_T(\psi,\varphi)$. Clear that $\varphi_T : \mathcal{B}(I\!R) \to I\!H$ is an additive function, and $(\psi, \varphi_{T_n}) \to 0$ for any decreasing sequence $\{T_n\}$ such that $\cap_{n\ge1} T_n = \emptyset$ because of (2.13). Therefore, φ_T is a vector-valued measure, and we write $\varphi_T = \int_T u(\tau)\, L(d\tau_+)\varphi$. Thus, we prove (2.14).

To prove (2.15) consider an identity

$$2 \sum_{j\in J_N(T)} u(\tau_j^{(N)})\, L(\tau_j^{(N)}, \tau_j^{(N)} + \delta_{N+1})\varphi = \sum_{j\in J_N(T)} u(\tau_j^{(N)})\, L(\tau_j^{(N)}, \tau_{j+1}^{(N)} + \delta_{N+1})\varphi +$$

$$+ \sum_{j\in J_N(T)} u(\tau_j^{(N)}) \Big(L(\tau_j^{(N)}, \tau_j^{(N)} + \delta_{N+1}) - L(\tau_j^{(N)} + \delta_{N+1}, \tau_{j+1}^{(N)}) \Big)\varphi.$$

As we noticed earlier, the first sum converges to integral (2.14). The second sum is absolutely bounded, and each term converges weakly to 0. Hence, the Lebesgue

theorem on dominated convergence implies a trivial limit for the second summand as $N \to \infty$. $\square$

Let $M^{(1)}, M^{(2)}, \ldots \in \mathcal{A}_{\mathcal{D}}$, for simplicity one can think that $M^{(k)}(T) = L_k \otimes M_k(T)$ (see (2.7)).

Definition 2.2 *Locally convex vector space $G_{\mathcal{D}} = G_{\mathcal{D}}(M^{(1)}, M^{(2)}, \ldots) \subseteq I\!\!H$ denotes a closure of* $\operatorname{Span} \mathcal{D} \subseteq I\!\!H$*, equipped by seminorms:*

$$\rho_t(\varphi) = ||\varphi|| + \sup_{\tau,\, \psi,\, \{i_k\},\, \{T_j\}} \frac{|(\psi, M^{(i_1)}(T_1) M^{(i_2)}(T_2)\, J_\tau\, \varphi)|}{\prod_j \, \mathrm{mes}\, T_j} \tag{2.16}$$

with $T_1 \cap T_2 = \emptyset,\ \psi \in I\!\!H_{T^a},\ ||\psi|| \leq 1,\ T_a = I\!\!R \setminus \{T_1 \cup T_2\},\ T_j \in (0,t),\ t \in I\!\!R_+$. We assume that $G_{\mathcal{D}}$ is dense in $I\!\!H$.

The group J_τ leaves invariant the set $G_{\mathcal{D}} : \rho_t(g) = \rho_t(J_\tau g),\ G_{\mathcal{D}} = J_\tau G_{\mathcal{D}}$. Cocycle property of coefficients (in the sense of paper [21]) is a simple consequence of the definition of $J_t : J_t^* A(T) J_t \psi(v) = (\widetilde{I}_T, e^{i\omega t} v)\psi(v) = (\widetilde{I}_{T+t}, v)\psi(v) = A(T+t)\psi(v),\ J_t^* A^+(T) J_t \psi(v) = A^+(T+t)\psi(v)$. This property of the coefficients $M(\cdot)$ of equation (2.10) generates a cocycle property of the solution $u(T)$. Later we will assume that the measures $M(\cdot) \in \mathcal{A}_{\mathcal{D}}$ poses the same property: $M(T+t) = J_t^*\, M(T)\, J_t$. It is clear that $G_{\mathcal{D}}$ is dense in $I\!\!H$ under assumptions of Example 2.1.

Theorem 2.1 *Let the interval adapted cocycle $u(t)$ be a bounded weak solution of (2.12), $\varphi \in \mathcal{D}$ and $L,\ ML,\ M^2 \in \mathcal{A}_{\mathcal{D}}$. If the semi-norm $\rho_\tau(L(0,\tau)\,\varphi)$ (2.16) is uniformly bounded for all $\tau \in I\!\!R_+$, then the adapted and the symmetric stochastic integrals satisfy the equation*

$$\int_T u(\tau) \Big(L(d\tau_+) + \frac{1}{2}\, M(d\tau_+) L(d\tau_+) \Big) \varphi = \int_T u(\tau)\, L(d\tau) \varphi \tag{2.17}$$

for every bounded set $T \in \mathcal{B}(I\!\!R)$.

Proof. Let $u(t)$ be a weak interval adapted solution of the equation (2.12). Consider a family of operators $u_s(t) = u(s) + u(s) M(s,t)$. The difference $\Delta_u(s,t) = u(t) - u_s(t)$ satisfies the equation $\Delta_u(s,t) = \int_s^t \int_s^\sigma \, u(\tau)\, M(d\tau_+)\, M(d\sigma_+)$ in the weak sense, and for every $\psi \in I\!\!H,\ \varphi \in \mathcal{D}$ there exist a uniform bound

$$|(\psi, \Delta_u(s,t)\, L(\tau_j^{(N)}, \tau_{j+1}^{(N)})\, \varphi)| \leq \frac{(t-s)^2}{2} \sup_{\tau \in T} ||u^*(\tau)\psi||\, \rho_{t-s}(L(0, 2^{-N})\, \varphi). \tag{2.18}$$

Since $L(\cdot)$ is an additive function, from the equation for $u(t)$ we have

$$u(\tau_j)\, L(\tau_j, \tau_{j+1})\varphi = \Big\{ u(\tau_{j+\frac{1}{2}}) L(\tau_j, \tau_{j+1}) - u(\tau_j) M(\tau_j, \tau_{j+\frac{1}{2}}) L(\tau_j, \tau_{j+\frac{1}{2}}) -$$

$$-u(\tau_j)\,M(\tau_j,\tau_{j+\frac{1}{2}})\,L(\tau_{j+\frac{1}{2}},\tau_{j+1})-\Delta_u(\tau_j,\tau_{j+\frac{1}{2}})L(\tau_j,\tau_{j+1})\Big\}\varphi. \tag{2.19}$$

For every $\psi\in I\!H,\ \varphi\in\mathcal{D}$ a prior bound follows from the definition (2.16)

$$\delta^{(1)}_{j,N}=|(\psi,u(\tau_j)\,M(\tau_j,\tau_{j+\frac{1}{2}})\,L(\tau_{j+\frac{1}{2}},\tau_{j+1})\,\varphi)|\le\tau_1^2\sup_{\tau\in(0,t)}||u^*(\tau)\psi||\,\rho_{\tau_1}(\varphi),$$

and from (2.12), (2.16) we obtain

$$\begin{aligned}\delta^{(2)}_{j,N}&=|(\psi,\Delta_u(\tau_j,\tau_{j+\frac{1}{2}})M(\tau_j,\tau_{j+1})L(\tau_j,\tau_{j+1})\,g)|\\&\le\tau_1^2\sup_{\tau\in(0,t)}||u^*(\tau)\psi||\,\rho_{\tau_1}(L(0,\tau_1))\,\varphi),\end{aligned}\tag{2.20}$$

where the semi-norm $\rho_{\tau_1}(L(0,\tau_1)\,\varphi)$ is uniformly bounded in N. Therefore,

$$\lim_N\sum_j\delta^{(1,2)}_{j,N}=0.$$

Now it is sufficient to observe that for operators $L, ML\in\mathcal{A}_{\mathcal{D}}$ the weak convergence in $I\!H$ take place by Lemma 2.1:

$$\lim_{N\to\infty}\sum_{J_N(T)}u(0,\tau_j)\,L(\tau_j,\tau_{j+1})\varphi=\int_T u(0,\tau)\,L(d\tau_+)\varphi,$$

$$\lim_{N\to\infty}\sum_{J_N(T)}u(0,\tau_j)\,M(\tau_j,\tau_{j+\frac{1}{2}})L(\tau_j,\tau_{j+\frac{1}{2}})\varphi=\frac{1}{2}\int_T u(0,\tau)\,M(d\tau_+)L(d\tau_+)\varphi.$$

Hence, the weak limit is well-defined on $\mathcal{D}$ by identity (2.19) and is called *a symmetric* (in the Stratonovich sense) stochastic integral:

$$\begin{aligned}\int_T u(\tau)\,L(d\tau)&=\lim_{N\to\infty}\sum_{j\in J_N(T)}u(\tau_{j+\frac{1}{2}})\,L(\tau_j,\tau_{j+1})\\&=\int_T u(\tau)\Big(L(d\tau_+)+\frac{1}{2}M(d\tau_+)L(d\tau_+)\Big).\end{aligned}\tag{2.21}$$

□

Equation (2.21) is satisfied whenever the measure $L(\cdot)$ of the symmetric integral satisfies the equation

$$L(T)+\frac{1}{2}\int_T M(dt_+)L(dt_+)=M(T)\qquad\forall T\in\mathcal{B}(I\!R)\tag{2.22}$$

with the given operator-valued measure $M(T)$ in the adapted integral. Assume that

$$M(T)=\sum_0^3 \mathcal{L}_i\otimes M_i(T),\quad L(T)=\sum_0^3 \ell_i\otimes M_i(T),$$

$$M_0(T)=I\cdot \operatorname{mes} T,\quad M_1(T)=A(T),\quad M_2(T)=A^+(T),\quad M_3(T)=\Lambda(T).$$

In this case the equation (2.22) has the form $\int_T \{L(dt)+\frac{1}{2}M(dt)L(dt)\}=M(T)$. Using the Ito multiplication table for stochastic differentials (see [2])

	dA^+	dA	$d\Lambda$	dt
dA^+	0	0	0	0
dA	dt	0	dA	0
$d\Lambda$	dA^+	0	$d\Lambda$	0
dt	0	0	0	0

in the left part of the equation $L(dt)+\frac{1}{2}M(dt)L(dt)=\sum \mathcal{L}_i\otimes M_i(T)$ we obtain a system of linear equations with respect to ℓ_k :

$$\mathcal{L}_0=\ell_0+\mathcal{L}_1\ell_2/2,\quad \mathcal{L}_1=\ell_1+\mathcal{L}_1\ell_3/2,\quad \mathcal{L}_2=\ell_2+\mathcal{L}_3\ell_2/2,\quad \mathcal{L}_3=\ell_3+\mathcal{L}_3\ell_3/2.$$

Assume that the operator $2+\mathcal{L}_3$ is invertible, and $(2+\mathcal{L}_3)^{-1}:\mathcal{H}\to \operatorname{dom}\mathcal{L}_1$. Then the solution of this system looks as follows: $\ell_3=2\mathcal{L}_3(2+\mathcal{L}_3)^{-1}$, $\ell_1=2\mathcal{L}_1(2+\mathcal{L}_3)^{-1}$, $\ell_2=2(2+\mathcal{L}_3)^{-1}\mathcal{L}_2$, $\ell_0=\mathcal{L}_0-\mathcal{L}_1(2+\mathcal{L}_3)^{-1}\mathcal{L}_2$. If we take

$$\mathcal{L}_0=iH+R^*\frac{1}{2-iK}R=-G,\quad \mathcal{L}_1=-L^*W,\quad \mathcal{L}_2=L,\quad \mathcal{L}_3=W-I,\tag{2.23}$$

as the coefficients $\{\mathcal{L}_k\}$ satisfying the condition necessary for existence of a unitary solution of (2.1) (see [2]), then $\ell_3=iK$, $\ell_1=iR^*$, $\ell_2=iR$, $\ell_0=iH$, where $K=2i\frac{I-W}{I+W}$, $R=-\frac{2i}{I+W}L$ are the coefficients of the symmetric operator $\widehat{H}(T)$:

$$i\widehat{H}(T)=\sum_{k=0}^3 \ell_k\otimes M_k(T),\quad M_k(T)=\int_T dt\, J_t^* M_k J_t\tag{2.24}$$

with $M_0=I$, $M_1=A$, $M_2=A^+$, $M_3=A^+A$.

Since K is a self–adjoint operator, there exist a unitary operator $W=(1+iK/2)(1-iK/2)^{-1}$ called *a Cayley transform* of $K/2$, and $2+\mathcal{L}_3=I+W=2(I-iK/2)^{-1}$ is an invertible operator. Furthermore, the operator $\mathcal{L}_1=iR^*(I-iK/2)^{-1}$ is well-defined provided the operators K, R and R^* have a joint dense domain D_0; the operators $\mathcal{L}_0=iH+R^*(2-iK)^{-1}R)$, $\mathcal{L}_2=(I-iK/2)^{-1}iR=-W\mathcal{L}_1^*$ are well-defined also. We call H, K, R, R^* *coefficients* of the interval-adapted measure

$M(T)=\int_T\, dt\, \sum \mathcal{L}_k \otimes J_t^* M_k J_t$, and the operators $\{-i\ell_k\}$ are called coefficients of the corresponding symmetric measure $\widehat{H}(T)=-i\int_T\, dt\, \sum \ell_k \otimes J_t^* M_k J_t$. The assumptions on coefficients of the symmetric measure $\widehat{H}(T)$ have a natural meaning: they are necessary for the measure $\widehat{H}(T)$ to be a densely defined symmetric operator; and the equation (2.23) implies an implicit form of the same assumption.

As we will see in the next Section, the relation between coefficients $\{\ell_k\}$ and $\{\mathcal{L}_k\}$ can be also derived from the strong resolvent limit for a family of the Schrödinger Hamiltonians. Since the conclusions of §2 look unusual, we consider the weak and the resolvent limits for an explicitly solvable model in Fock space. By this reason we restrict ourself to the case of Hamiltonians with commuting coefficients.

3 The strong resolvent limit of the Schrödinger Hamiltonians

The weak and the resolvent limits of generators of strongly continuous unitary groups may be different in a quite elementary case. The next example gives a hint to our main result. Consider a family of unitary groups $\exp\{itH_\alpha\}=U_t^{(\alpha)}$:

$$U_t^{(\alpha)}\psi(x)=\psi(x-t)\exp\Big\{i\lambda\int_0^t d\tau\, V_\alpha(x-t+\tau)\Big\},\quad x,\lambda\in I\!R,\quad \psi\in\mathcal{L}_2(I\!R)$$

where $V_\alpha(x)=(2\pi\alpha)^{-1/2}\exp\{-x^2/2\alpha\},\ \alpha\in I\!R_+$. Clearly $V_\alpha(x)\to\delta(x)$ as $\alpha\to+0$, and $\int_0^t d\tau\, V_\alpha(x-t+\tau)\to I_{[0,t)}(x)$, with an indicator function $I_\Gamma(x)$ of the borelean set Γ. Therefore, *the weak limit* of the family of essentially self-adjoint operators $H_\alpha=i\partial_x+\lambda V_\alpha(x)$ generates a well-defined bilinear form on $W_2^1(I\!R)$

$$H_*[\varphi,\psi]=(\varphi,\widehat{H}_w\psi)=i(\varphi,\psi')+\lambda\overline{\varphi}(0)\psi(0)$$

that corresponds to a formal operator $\widehat{H}=i\partial_x+\lambda\delta(x)$.

On the other hand, *the strong limit* of the family of unitary groups $U_t^{(\alpha)}$

$$U_t\psi(x)=e^{it\widehat{\mathbf{H}}}=\lim_{\alpha\to+0}U_t^{(\alpha)}\psi(x)=\psi(x-t)e^{i\lambda I_{[0,t)}(x)}=\psi(x-t)\Big\{(e^{i\lambda}-1)I_{[0,t)}(x)+1\Big\}$$

implies *the strong resolvent limit* $\widehat{\mathbf{H}}$ which can be described by a formal generator

$$\widehat{\mathbf{H}}=r-\lim H_\alpha=i\partial_x+i(e^{i\lambda}-1)\delta(x)$$

or by bilinear form

$$\mathbf{H}_*[\varphi,\psi]=\lim_{t\to0}\frac{d}{dt}(\varphi,U_t\psi)=i(\varphi,\psi')+i(e^{i\lambda}-1)\overline{\varphi}(0)\psi(0).$$

Important observation is that $\widehat{\mathbf{H}} = r - \lim H_\alpha \neq \widehat{H} = w - \lim H_\alpha$. The multiplier $e^{i\lambda} - 1$, an analog of the factor $W - I = \mathcal{L}_3$ in equation (2.23), appears in a manifest form in the equation for $\widehat{\mathbf{H}}$. A similar phenomena happen in what follows.

The range of the resolvent of the limit unitary group U_t is a natural domain for the generator $\widehat{\mathbf{H}}$. In this example it can be described explicitly by an equation:

$$R_\mu \psi(x) = \int_0^\infty dt\, e^{-\mu t}\psi(x-t) + \theta(x)(e^{i\lambda}-1)e^{-\mu x}\int_0^\infty dt\, e^{-\mu t}\psi(-t),$$

where $\theta(x)$ is an indicator function of the half–line $I\!\!R_+$. The structure of the resolvent shows that the functions from the domain of its generator $\widehat{\mathbf{H}}$ have phase jumps at the origin $x = 0 : \lim_{x\to+0} R_\mu\psi(x) = e^{i\lambda}\lim_{x\to-0} R_\mu\psi(x)$. Hence the domain $\mathcal{D}_\lambda$ of $\widehat{\mathbf{H}}$ consists of functions with a standard discontinuity at the origin:

$$\psi :\ \psi \in W_2^1(I\!\!R \setminus \{0\}), \quad \lim_{x\to+0}\psi(x) = e^{i\lambda}\lim_{x\to-0}\psi(x)$$

and the operator $\widehat{\mathbf{H}}$ acts as $i\partial_x$ if $x \neq 0$. The left and the right limits exist at the origin for every function from $\mathcal{D}_\lambda$ because of the imbedding $W_2^1(I\!\!R\setminus\{0\}) \subset C(I\!\!R\setminus\{0\})$. Integration by parts and the identity $\overline{\phi(x)}\psi(x)|_{+0}^{-0} = 0$ for functions from $\mathcal{D}_\lambda$ proves that the operator $\widehat{\mathbf{H}}$ is symmetric. The existence of a solution from $\mathcal{D}_\lambda$ for the problem $(\widehat{\mathbf{H}} + i\mu)\psi(x) = f(x),\ x \neq 0$ with the boundary condition as above for every $f \in \mathcal{L}_2(I\!\!R),\ \mu = \pm 1$ implies the self-adjoint property of $\widehat{\mathbf{H}}$.

Consider the Schrödinger equation $\partial_t\psi_t = iH_\alpha\psi_t$ with the self-adjoint Hamiltonian $H_\alpha = H\otimes I + I\otimes\widehat{\mathbf{E}} + H_{int}^{(\alpha)}$ depending on a scaling parameter $\alpha \in (0,1]$:

$$H_{int}^{(\alpha)} = K\otimes A^+(g_\alpha)A(g_\alpha) + R\otimes A^+(f_\alpha) + R^*\otimes A(f_\alpha). \tag{3.1}$$

Assume that coefficients $H,\, K,\, R \in \mathcal{C}(\mathcal{H})$ have a joint spectral family E_λ :

$$H = \int \nu_\lambda\, dE_\lambda, \quad K = \int \lambda\, dE_\lambda, \quad R = \int \rho_\lambda e^{i\Phi_\lambda}\, dE_\lambda,$$

where $\nu,\ \rho,\ \Phi$ are measurable real functions corresponding to operators H, K, and R; and functions $f_\alpha(\omega) = f(\alpha\omega)$, $g_\alpha = g(\alpha\omega)$ f, g are elements of the space $L^+_{2,\widetilde{1}}(I\!\!R) = \{f :\ f \in L_2(I\!\!R),\ \widetilde{f} = \mathcal{F}^{-1}_{\omega\to t}f \in L_1^+(I\!\!R)\},\ f(0) = g(0) = (2\pi)^{-1/2}$. We omit the indices α, when there is no conflicts.

Denote by $\widehat{P}_t = \widehat{P}_t^{(\alpha)}(\lambda)$ a one-parameter unitary group in $L_2(I\!\!R)$ with the generator $\widehat{N}_\alpha(\lambda) = \omega + \lambda|g_\alpha\rangle\langle g_\alpha|$ and prove that the solution of the Schrödinger equation $U_t^{(\alpha)} = \exp\{iH_\alpha t\}$ acts as follows

$$U_t^{(\alpha)} h\otimes\psi(v) = \int e^{i\nu_\lambda t}\, dE_\lambda\, h\otimes\psi\Big(\widehat{P}_t(\lambda)v + i\rho_\lambda e^{i\Phi_\lambda}\int_0^t \widehat{P}_s(\lambda) f_\alpha\, ds\Big)$$

$$\times \exp\Big\{ i\rho_\lambda e^{-i\Phi_\lambda} \int_0^t (f_\alpha, \widehat{P}_s(\lambda) v)\, ds - \rho_\lambda^2 \int_0^t dr \int_0^r ds\, (f_\alpha, \widehat{P}_{r-s}(\lambda) f_\alpha) \Big\}. \tag{3.2}$$

For any bounded weakly measurable function $\widehat{P}_t : I\!\!R \to \mathcal{B}(\mathcal{H})$ the following equations are consequences of linear changes of variables:

$$\begin{aligned}
\int_0^t \widehat{P}_r\, dr \quad &+ \quad \int_0^s \widehat{P}_{r+t}\, dr = \int_0^{t+s} \widehat{P}_r\, dr, \\
\Big(\int_0^t \int_0^r \quad &+ \quad \int_0^s \int_0^r \Big) \widehat{P}_{r-p}\, dr\, dp + \int_0^t \int_0^s \widehat{P}_{r+p}\, dr\, dp \\
= \Big(\int_0^t \int_0^r \quad &+ \quad \int_t^{t+s} \int_t^{t+r} + \int_0^t \int_t^{t+s} \Big) \widehat{P}_{r-p}\, dr\, dp = \int_0^{t+s} \int_0^r \widehat{P}_{r-p}\, dr\, dp,
\end{aligned}$$

which are related to a group property of $U_t^{(\alpha)}$: the first equation generates the group property of the argument of the coherent vector, and the second one generates the same property of the normalizing factor in (3.2).

The weak form of the Schrödinger equation can be verified by differentiation of the quadratic form $\varphi_t = (h \otimes \psi(v), U_t^{(\alpha)}\, h \otimes \psi(v))$:

$$\varphi_t = \int e^{i\nu_\lambda t} (dE_\lambda h, h) \exp\Big\{ \Big(v, \widehat{P}_t(\lambda) v + i\rho_\lambda e^{i\Phi_\lambda} \int_0^t \widehat{P}_s(\lambda) f_\alpha\, ds \Big) +$$

$$+ i\rho_\lambda e^{-i\Phi_\lambda} \int_0^t (f_\alpha, \widehat{P}_s(\lambda) v)\, ds - \rho_\lambda^2 \int_0^t dr \int_0^r (f_\alpha, \widehat{P}_{r-s}(\lambda) f_\alpha)\, ds \Big\}.$$

We have clearly

$$\begin{aligned}
-i\varphi_t' \Big|_{t=0} \quad &= \quad e^{||v||^2} \int (dE_\lambda h, h) \Big(\nu_\lambda + \lambda |(g_\alpha, v)|^2 \\
&+ \quad 2\,\mathrm{Re}\, \rho_\lambda e^{i\Phi_\lambda} (v, f_\alpha) + \int d\omega\, \omega |v(\omega)|^2 \Big) = (h \otimes \psi(v), H_\alpha h \otimes \psi(v)).
\end{aligned}$$

The one-parameter group $U_t^{(\alpha)}$ is unitary because of basic properties of spectral operators E_λ and the equation for the norm of an exponential vector:

$$\Big\| \psi \Big(\widehat{P}_t v + i\rho e^{i\Phi} \int_0^t \widehat{P}_s f\, ds \Big) \Big\|^2 =$$

$$= \exp\Big\{ ||v||^2 + \rho^2 \Big\| \int_0^t \widehat{P}_s f\, ds \Big\|^2 - 2\rho\, \mathrm{Im}\, e^{-i\Phi} \int_0^t (f, \widehat{P}_s v)\, ds \Big\}. \tag{3.3}$$

Since $\widehat{P}_t^{(\alpha)}$ is a unitary group, we have

$$2\mathrm{Re} \int_0^t dr \int_0^r ds\, \widehat{P}_{r-s} = \overline{\int_0^t dr\, \widehat{P}_s} \int_0^t ds\, \widehat{P}_r.$$

Hence, all terms in (3.3) besides $||v||^2$ are canceled out by the corresponding terms in the normalizing factor. Thus, we obtain
Theorem 3.1 *The unitary group of operators $U_t^{(\alpha)} = \exp\{itH_\alpha\}$ acts as follows*

$$U_t^{(\alpha)} h \otimes \psi(v) = \int e^{i\nu_\lambda t} dE_\lambda\, h \otimes \psi\Big(\widehat{P}_t^{(\alpha)}(\lambda)v + i\rho_\lambda e^{i\Phi_\lambda} \int_0^t \widehat{P}_s^{(\alpha)}(\lambda) f_\alpha\, ds\Big)\times$$

$$\times \exp\Big\{i\rho_\lambda e^{-i\Phi_\lambda} \int_0^t (f_\alpha, \widehat{P}_s^{(\alpha)}(\lambda)v)\, ds - \rho_\lambda^2 \int_0^t dr \int_0^r (f_\alpha, \widehat{P}_{r-s}^{(\alpha)}(\lambda) f_\alpha)\, ds\Big\}. \qquad (3.4)$$

Equation (3.4) contains four terms depending on α in the right hand side. The next Lemma "On Four Limits" (see [20]) justifies the limit of (3.4) as $\alpha \to 0$.

Lemma 3.1 *Let $\widehat{P}_t^{(\alpha)}(\lambda)$ be one-parameter unitary group in $L_2(I\!R)$ generated by $\widehat{N}_\alpha(\lambda)$. Then for f, $g \in L^+_{2,\widetilde{1}}(I\!R)$ such that $f(0) = g(0) = \frac{1}{\sqrt{2\pi}}$ there exist limits:*

(1) $\int_0^t ds (g_\alpha, \widehat{P}_s^{(\alpha)}(\lambda) f_\alpha) \to (2 - i\lambda)^{-1}$;

(2) $\int_0^t ds\, \widehat{P}_s^{(\alpha)}(\lambda) f_\alpha(\omega) \to e^{i\omega t} \widetilde{I}_{(0,t)}(\omega)(1 - i\lambda/2)^{-1}$;

(3) $(g_\alpha, \widehat{P}_t^{(\alpha)}(\lambda)v) \to (1 - i\lambda/2)^{-1} \mathcal{F}^*_{\omega\to t} v$;

(4) $\widehat{P}_t^{(\alpha)}(\lambda) \to \exp\{iZ(\lambda)\widehat{\pi}_{(0,t)}\} = \widehat{P}_t(\lambda), \quad \exp\{iZ(\lambda)\} = (2 + i\lambda)/(2 - i\lambda)$,

with $\widehat{\pi}_T$ a projector in $L_2(I\!R)$: $\widehat{\pi}_T = \mathcal{F}^*_{t\to\omega} I_T(t) \mathcal{F}_{\omega\to t}$. Limits (2), (3) *are weak in $L_2(I\!R)$, the limit* (4) *is strong in $L_2(I\!R)$.*

Proof. Limits (1-3) can be derived from the Taylor decomposition of the operator $\widehat{P}_t^{(\alpha)}(\lambda)$ in a power series $\sum C_k(\alpha, t)\lambda^k$ (as in [3, 7]) and sequential limiting procedure in each term as $\alpha \to 0$. This operation maps the series into a geometrical progression that converges absolutely for $|\lambda| < 2$. The Fatou–Privalov theorem enables to extend this result to the entire real line $\lambda \in I\!R$, provided there exist a uniform absolute bound for the series in the upper half-plain $\mathrm{Im}\lambda \geq 0$. We can prove the required bound for the series corresponding to limit (4), but we do not have such bounds to prove limits (1)-(3). The method used below consists in application of properties of completely monotone functions (see [22], Chap. XIII, §4) and the Duhamel equation for the group $\widehat{P}_t^{(\alpha)}(\lambda)$:

$$\widehat{P}_t^{(\alpha)}(\lambda) = e^{i\omega t} + i\lambda \int_0^t ds\, e^{i\omega(t-s)} |g_\alpha\rangle\langle g_\alpha|\, \widehat{P}_s^{(\alpha)}(\lambda). \qquad (3.5)$$

(1) Set $a_t = a_t^{(\alpha)}(\lambda) = (g_\alpha, \widehat{P}_t^{(\alpha)}(\lambda) f_\alpha)$. As a consequence of (3.5), we have

$$\int_0^t a_s\, ds = \Phi_t(g_\alpha, f_\alpha) + i\lambda \int_0^t \Phi_{t-s}(g_\alpha, g_\alpha)\, a_s\, ds, \tag{3.6}$$

with $\Phi_t(g,f) = \int_0^t ds\, (g, e^{i\omega s} f) = \mu_{f,g}(0,t)$, where $\mu_{f,g}(\cdot)$ is2 the probability measure such that

$$\mu_{f,g}(T) = \int_T ds \int_{I\!R} dt\, \overline{\widetilde{g}(s-t)} \widetilde{f}(t), \quad \mu_{f,g}(I\!R) = 2\pi g(0) f(0) = 1.$$

Denote by $\widetilde{a}_p = \widetilde{a}_p^{(\alpha)}(\lambda)$ the Laplace transform of the bounded continuous function a_t. Since the equation (3.6) contains a convolution, we obtain an equivalent algebraic equation $p^{-1}\widetilde{a}_p = \widetilde{\Phi}_p(g,f) + i\lambda \widetilde{a}_p \widetilde{\Phi}_p(g,g)$. The solution reads as

$$\widetilde{a}_p = p\widetilde{\Phi}_p(g,f)(1 - i\lambda p \widetilde{\Phi}_p(g,g))^{-1},$$

where $p\widetilde{\Phi}_p(g,f) = \int_0^\infty e^{-pt} d\,(g, e^{i\omega t} f) = \widetilde{\mu}_{f,g}(p)$ is the Laplace transform of the probability measure $\mu_{f,g}$. Since $\Phi_t(g_\alpha, f_\alpha) = \mu_{f,g}(0, t/\alpha)$, then $\widetilde{a}_p^{(\alpha)}(\lambda) = \widetilde{a}_{p\alpha}(\lambda)$ and

$$\widetilde{a}_p(\lambda) = \frac{\widetilde{\mu}_{f,g}(p)}{1 - i\lambda \widetilde{\mu}_{g,g}(p)} = \frac{\widetilde{\mu}_{f,g}(p)}{1 + |\lambda|^2 \widetilde{\mu}_{g,g}^2(p)} + i\overline{\lambda} \frac{\widetilde{\mu}_{f,g}(p)\widetilde{\mu}_{g,g}(p)}{1 + |\lambda|^2 \widetilde{\mu}_{g,g}^2(p)}.$$

We recall that the class of the Laplace transforms of positive measures coincides with the class of completely positive functions. It contains the function $(1 + |\lambda|^2 p)^{-1}$, and it is closed w.r.t. addition, multiplication and composition (see. [22], Chap. XIII, §4). Hence, the real and imaginary parts of $\widetilde{a}_{p\alpha}(\lambda)$ are the Laplace transforms of bounded measures:

$$\widetilde{a}_p^{(\alpha)}(\lambda) = \int_0^\infty e^{-pt} \Big(A_\lambda(\alpha^{-1} dt) + i\lambda B_\lambda(\alpha^{-1} dt) \Big),$$

$$A_\lambda(I\!R_+) = \frac{\mu_{f,g}(I\!R_+)}{1 + |\lambda|^2 \mu_{g,g}^2(I\!R_+)} = \frac{2}{4+\lambda^2}, \quad B_\lambda(I\!R_+) = \frac{\mu_{f,g}(I\!R_+)\mu_{g,g}(I\!R_+)}{1 + |\lambda|^2 \mu_{g,g}^2(I\!R_+)} = \frac{1}{4+\lambda^2}$$

It follows from here for any $\lambda \in C\!\!\!\!I$ that

$$\int_T a_s^{(\alpha)}(\lambda)\, ds = A_\lambda(T/\alpha) + i\lambda B_\lambda(T/\alpha) \qquad \forall T \in \mathcal{B}(I\!R_+) \tag{3.7}$$

$$\lim_{\alpha \to +0} \int_0^t a_s^{(\alpha)}(\lambda)\, ds = A_\lambda(I\!R_+) + i\lambda B_\lambda(I\!R_+) = (2 - i\lambda)^{-1}.$$

(2) Denote $\beta_t^{(\alpha)}(\lambda, \omega) = \int_0^t ds\, \widehat{P}_s^{(\alpha)}(\lambda) f_\alpha(\omega)$, and for $v \in L_2(I\!R)$ set

$$b_t^{(\alpha)}(\lambda, v) = \int d\omega\, \overline{v}(\omega) \beta_t^{(\alpha)}(\lambda, \omega), \quad \varphi_t(v, f) = \int_0^t ds\, (v, e^{i\omega s} f).$$

From equations (3.5) and (3.7) follows an identity

$$b_t^{(\alpha)} = \varphi_t(v, f_\alpha) + i\lambda \int_0^t ds\, \varphi_{t-s}(v, g_\alpha) \int_0^{t-s} d\tau\, (g_\alpha, \widehat{P}_\tau^{(\alpha)}(\lambda) f_\alpha) =$$

$$= \varphi_t(v, f_\alpha) + i\lambda \int_0^t ds\, \varphi_{t-s}(v, g_\alpha) \Big(A_\lambda(\alpha^{-1}(t-s)) + i\lambda B_\lambda(\alpha^{-1}(t-s)) \Big). \qquad (3.8)$$

The integral $\int_T dt e^{i\omega t} g_\alpha(\omega)$ converges in $L_2(I\!R)$ to $e^{i\omega t}\widetilde{I}_{(0,t)}(\omega)$ as $\alpha \to +0$. Since $L_2(I\!R) \in L_1^{loc}(I\!R)$, the functions $\varphi_t(v, f_\alpha)$ and $\varphi_t(v, g_\alpha)$ converge to $(v, e^{i\omega t}\widetilde{I}_{(0,t)})$. The function $A_\lambda(\alpha^{-1}t) + i\lambda B_\lambda(\alpha^{-1}t)$ is measurable in t, bounded uniformly in $\alpha \in (0,1]$ and converges at each point to $(2-i\lambda)^{-1}$ as $\alpha \to 0$. Therefore, it is possible to pass to the limit in (3.8): $b_p^{(\alpha)}(\lambda, v) \to (1 + i\lambda/(2-i\lambda))(v, e^{i\omega t}\widetilde{I}_{(0,t)})$. Hence, for every $v \in L_2(I\!R)$

$$\lim_{\alpha\to 0} \int_0^t ds\, (v, \widehat{P}_s^{(\alpha)}(\lambda) f_\alpha) = (1 - i\lambda/2)^{-1} (v, e^{i\omega t}\widetilde{I}_{(0,t)}), \qquad (3.9)$$

that is $w - \lim \beta_t^{(\alpha)}(\lambda, \omega) = (1 - i\lambda/2)^{-1} e^{i\omega t}\widetilde{I}_{(0,t)}(\omega)$.

(3) Rewrite (3.9) in an equivalent form:

$$\lim_{\alpha\to 0} \int_0^t ds\, (v, \widehat{P}_s^{(\alpha)}(\lambda) f_\alpha) = \frac{2}{2-i\lambda} \int \overline{\mathcal{F}_{\omega\to\tau} v}\, I_{(0,t)}(\tau)\, d\tau. \qquad (3.10)$$

Since $(g_\alpha, P_t^{(\alpha)}(\lambda)v) = \overline{(v, P_{-t}^{(\alpha)}(\lambda) g_\alpha)}$, from (3.10) follows an equation

$$\lim_{\alpha\to 0} \int_0^t ds\, (g_\alpha, \widehat{P}_s^{(\alpha)}(\lambda) v) = \overline{\frac{2}{2+i\lambda} (\mathcal{F}^*_{\omega\to\tau} v, I_{(0,t)}(\tau))} = \frac{2}{2-i\lambda} (I_{(0,t)}(\tau), \mathcal{F}^*_{\omega\to\tau} v). \qquad (3.11)$$

The set of the indicator functions of bounded Borelean subsets in $I\!R$ is total in $L_2(I\!R)$. Therefore, from (3.11) follows

$$\lim_{\alpha\to 0} \int ds\, \overline{r}(t) (g_\alpha, \widehat{P}_s^{(\alpha)}(\lambda) v) = \frac{2}{2-i\lambda} (r(\cdot), \mathcal{F}^*_{\omega\to\cdot} v)$$

for every $r, v \in L_2(I\!R)$, or $w - \lim (g_\alpha, \widehat{P}_t^{(\alpha)}(\lambda)v) = (1 - i\lambda/2)^{-1} \mathcal{F}^*_{\omega\to t} v$ in $L_2(I\!R)$. From here we conclude that the norm of the weakly converging family

$$g_{\alpha,t} = (g_\alpha, \widehat{P}_t^{(\alpha)}(\lambda)v) \in L_2(I\!R)$$

is uniformly bounded for every $\alpha \in (0,1]$.

(4) Note that the strong convergence of unitary operators follows from the weak convergence, and the weak convergence follows from the convergence ot corresponding quadratic forms. Hence, it suffices to prove that $(v, \widehat{P}_t^{(\alpha)}(\lambda)v) \to (v, \widehat{P}_t(\lambda)v)$

$\forall v \in L_2(I\!R)$. Denote $c_t^{(\alpha)} = c_t^{(\alpha)}(\lambda, v) = (v, \widehat{P}_t^{(\alpha)}(\lambda)v)$. From (3.1) follows the identity:

$$c_t^{(\alpha)} = (v, e^{i\omega t}v) + i\lambda \int_0^t ds\,(v, e^{i\omega(t-s)}g_\alpha)(g_\alpha, \widehat{P}_s^{(\alpha)}(\lambda)v), \tag{3.12}$$

where the sequence $g_{\alpha,s} = (g_\alpha, \widehat{P}_s^{(\alpha)}(\lambda)v)$ is uniformly bounded and weakly converging in $L_2(I\!R)$. Set $\widetilde{v}(t) = \mathcal{F}_{\omega\to t}v(\omega)$, and $\widetilde{v}_\alpha(t) = \overline{(v, e^{i\omega t}g_\alpha)}$. Since the function $g(\omega) = \mathcal{F}_{t\to\omega}\widetilde{g}(t)$ can be decomposed into an integral, the square of the norm of $\delta_\alpha = ||\widetilde{v} - \widetilde{v}_\alpha||$ can be represented as an integral of absolutely integrable functions

$$\delta_\alpha^2 = \int_{I\!R^3} dt\,dr\,ds\,(\overline{\widetilde{v}(t-\alpha s)} - \overline{\widetilde{v}(t)})\,\overline{\widetilde{g}(s)}\,(\widetilde{v}(t-\alpha r) - \widetilde{v}(t))\,\widetilde{g}(r).$$

Clearly, $\delta_\alpha \to 0$ because it is possible to pass to the limit in this integral as $\alpha \to 0$ for every $v \in L_2(I\!R)$, $g \in L^+_{2,\widetilde{1}}(I\!R)$. Therefore, we prove the norm convergence $\widetilde{v}_\alpha \to \widetilde{v}$. Now we can apply the equation (3) to the weakly converging sequence $(g_\alpha, \widetilde{P}_s^{(\alpha)}(\lambda)v)$ and pass to the limit in the integral (3.12):

$$c_t^{(\alpha)} \to \Big(v, e^{i\omega t}\{I + \frac{2i\lambda}{2-i\lambda}\int_0^t ds\, e^{-i\omega s}\mathcal{F}^*_{\omega\to s}\}v\Big) = \Big(v, e^{i\omega t}\{I + \frac{2i\lambda}{2-i\lambda}\widehat{\pi}_{(0,t)}\}v\Big).$$

In this way we obtain the weak and the strong limits

$$\lim_{\alpha\to 0}\widehat{P}_t^{(\alpha)}(\lambda) = I + \frac{2i\lambda}{2-i\lambda}\widehat{\pi}_{(0,t)} = \frac{2+i\lambda}{2-i\lambda}\widehat{\pi}_{(0,t)} + \widehat{\pi}^a_{(0,t)} = \exp\{iZ(\lambda)\widehat{\pi}_{(0,t)}\},$$

with the projector-valued family $\widehat{\pi}_T$ from $\mathcal{B}(L_2(I\!R))$, and $I - \widehat{\pi}_T = \widehat{\pi}_{T^a}$.

□

Limits (1)-(4) describe what occur with the solution (3.3) as $\alpha \to 0$. Substituting (1)-(4) to (3.3) we obtain the unitary group $U_t = \exp\{i\widehat{\mathbf{H}}t\} = s - \lim U_t^{(\alpha)}$:

$$\begin{aligned} U_t\,h\otimes\psi(v) &= \int e^{-G_\lambda t}\,dE_\lambda\,h\otimes\psi\Big(e^{iZ(\lambda)\widehat{\pi}_{(0,t)}}e^{i\omega t}v + i\rho_\lambda e^{i\Phi_\lambda}\frac{2}{2-i\lambda}\widetilde{I}_{(0,t)}\Big)\\ &\times\ \exp\Big\{i\rho_\lambda e^{-i\Phi_\lambda}\frac{2}{2-i\lambda}(\widetilde{I}_{(0,t)}, e^{i\omega t}v)\Big\}, \end{aligned}\tag{3.13}$$

with $G_\lambda = -i\nu_\lambda - \rho_\lambda^2/(2-i\lambda)$. The group property of U_t follows from (3.13) and from the properties of the Fourier transform:

$$e^{i\omega t}\widehat{\pi}_T e^{-i\omega t} = \widehat{\pi}_{T+t}, \quad e^{i\omega t}\widetilde{I}_{(0,t)}(\omega) = \overline{\widetilde{I}}_{(0,t)}(\omega).$$

Taking the time derivative

$$\lim_{t\to+0}\frac{1}{i}\frac{d}{dt}(g\otimes\psi(f), U_t\,h\otimes\psi(v)) = \mathbf{H}_*[g\otimes\psi(f), h\otimes\psi(v)]$$

where $g,h\in D\subseteq\mathcal{H},\quad f,v\in\widetilde{W}_2^1(I\!R)$ we obtain a bilinear form

$$\mathbf{H}_*[g\otimes\psi(f),h\otimes\psi(v)]=e^{(f,v)_{L_2}}\Big((g,H_0h)_{\mathcal{H}}+(g,H_1h)_{\mathcal{H}}\,\widetilde{v}(0)+(g,H_2h)_{\mathcal{H}}\,\overline{\widetilde{f}(0)}$$

$$+(g,h)_{\mathcal{H}}\int d\omega\overline{f}(\omega)g(\omega)\,\omega+(g,H_3h)_{\mathcal{H}}\,\overline{\widetilde{f}(0)}\widetilde{v}(0)\Big),$$

where

$$H_0=H-R^*\frac{K}{4+K^2}R+R^*\frac{2i}{4+K^2}R=iG,\quad H_1=R^*\frac{2}{2-iK},$$

$$H_2=\frac{2}{2-iK}R,\quad H_3=\frac{2K}{2-iK}=i(I-W),\quad W=\frac{2+iK}{2-iK}.\tag{3.14}$$

In what follows we assume that the operator $G=-iH+\frac{i}{4}L^*KL+\frac{1}{2}L^*L$ is a generator of one-paremeter contraction semigroup $W_t=\exp\{-Gt\}$ in $\mathcal{H}$ such that

$$D=dom\,H\cap dom\,L^*L\subseteq dom\,G\subseteq dom\,L,\quad G^*\phi+G\phi=L^*L\phi\quad\forall\phi\in D,$$

$H_s=-H+\frac{1}{4}L^*KL$ is a operator symmetric on D, and D is core for G (see [23]).

The formal operator expression describing the quadratic form $\mathbf{H}_*$ reads as $\widehat{\mathbf{H}}=I\otimes\widehat{E}+H_0\otimes I+(2\pi)^{-1/2}H_1\otimes A(1)+(2\pi)^{-1/2}H_2\otimes A^+(1)+(2\pi)^{-1}H_3\otimes A^+(1)A(1)$.

Consider the function of a set $u(s,t)=J_sU_{t-s}J_t^*$. Since $J_t^*\psi(v)=\psi(e^{-i\omega t}v)$, we obtain from (3.13)

$$u(T)\,h\otimes\psi(v)=\int e^{-G_\lambda\mathrm{mes}\,T}\,dE_\lambda\,h\otimes\psi\Big(e^{iZ(\lambda)\widehat{\pi}_T}v+$$

$$+i\rho_\lambda e^{i\Phi_\lambda}\frac{2}{2-i\lambda}\widetilde{I}_T\Big)\exp\Big\{i\rho_\lambda e^{-i\Phi_\lambda}\frac{2}{2-i\lambda}(\widetilde{I}_T,v)\Big\}.$$

The $u(T)$ family of operators is interval adapted, commutative for disjoint arguments, and satisfies the cocycle composition rule $u(T_1\cup T_2)=u(T_1)u(T_2),\quad T_1\cap T_2=\emptyset$. The weak evolution equation for $u(T)$ reads

$$d(h\otimes\psi(v),u(0,t)h\otimes\psi(v))=i(h\otimes\psi(v),u(0,t)H(dt_+)h\otimes\psi(v)),$$

where $i\widehat{H}(T)=M(T)=i\int_T dt(J_t\widehat{\mathbf{H}}J_t^*-\widehat{\mathbf{E}}\otimes I)$ is well-defined operator-valued measure:

$$\begin{aligned}\widehat{H}(T)\ &=\ \Big(H-R^*\frac{K}{4+K^2}R+R^*\frac{2i}{4+K^2}R\Big)\otimes\mathrm{mes}\,T\\ &+\ \frac{2}{2-iK}R\otimes A^+(T)+R^*\frac{2}{2-iK}\otimes A(T)+i(I-W)\otimes\Lambda(T).\end{aligned}$$

Hence, we obtain the following result.

Theorem 3.2 *The family of solutions for the Schrödinger equation with Hamiltonian (3.1) converges in $L_2(I\!R)$ to the solution of stochastic differential equation (2.10) with coefficients (2.23): $u(0,t) = s - \lim_{\alpha\to 0} U_t^{(\alpha)} J_t^*$. Conditions (3.14) are necessary for the generator $\widehat{\mathbf{H}}$ of the limit unitary group U_t to be symmetric.*

4 Surprises of the resolvent

Consider a Fock vector $\Phi \in \mathbf{h} = \mathcal{H} \otimes \Gamma^S(\mathcal{L}_2(I\!R))$ belonging to the range of the resolvent

$$\Phi = R_\mu h \otimes \psi(v) = \int_0^\infty dt\, e^{-\mu t} U_t\, h \otimes \psi(v) = \{\Phi_n(\omega)\},$$

$$\Phi_n(\cdot) : I\!R^n \to \mathcal{H}, \quad \omega = \{\omega_1, \ldots, \omega_n\}$$

with components (3.13):

$$\begin{aligned} \Phi_n(\omega) &= \int_0^\infty dt\, \exp\Big\{-\mu(G+t) - L^* W \int_0^t \widetilde{v}(-\tau)\, d\tau\Big\} \phi_{n,t}(\omega), \\ \phi_{n,t}(\omega) &= \prod_1^n \Big((W-1)\pi_{[0,t)} e^{i\omega_k t} v(\omega_k) + e^{i\omega_k t} v(\omega_k) + L\widetilde{I}_{[0,t)}(\omega_k)\Big) h, \end{aligned}$$

for commuting operators L, W, and G as above:

$$L = \int dE_\lambda\, L(\lambda), \quad W = \int dE_\lambda\, W(\lambda), \quad G = \int dE_\lambda\, G(\lambda),$$

$L(\lambda) = 2i\rho(\lambda) e^{-i\Phi(\lambda)}(2 - i\lambda)^{-1}, \quad W(\lambda) = e^{iZ(\lambda)}, \quad G(\lambda) = -i\nu(\lambda) + \rho(\lambda)^2/(2 - i\lambda).$

Denote by $\widetilde{\phi}_{n,t}$ the Fourier transform of the function $\phi_{n,t}(\omega)$ with respect to variables $\omega = \{\omega_1, \ldots, \omega_n\}$:

$$\widetilde{\phi}_{n,t}(\tau) = \prod_1^n \Big((W - I) I_{[0,t)}(\tau_k) \widetilde{v}(\tau_k - t) + \widetilde{v}(\tau_k - t) + L\, I_{[0,t)}(\tau_k)\Big) h,$$

where $\tau = \{\tau_1, \ldots, \tau_n\}$. Let $\mathcal{K}$ be a subset in $\{1, \ldots, n\}$ and let $\mathcal{K}^a$ be its complement. Put

$$P_{\mathcal{K},t}^{(n)}(\tau) = \prod_{k\in\mathcal{K}} \Big((W - I)\widetilde{v}(\tau_k - t) + L\Big) I_{[0,t)}(\tau_k) \in \mathcal{B}(\mathbf{h}).$$

then

$$\widetilde{\phi}_{n,t}(\tau) = \sum_{\mathcal{K}} \Big(P_{\mathcal{K},t}^{(n)}(\tau) \prod_{m\in\mathcal{K}^a} \widetilde{v}(\tau_m - t)\Big) h. \tag{4.1}$$

The functions $P^{(n)}_{\mathcal{K},t}(\tau)$ have discontinuity in hyperplanes where variables τ_k change the sign:

$$\lim_{\tau_k\to -0} P^{(n)}_{\mathcal{K},t}(\tau) = I_{\mathcal{K}^a}(k) P^{(n)}_{\mathcal{K},t}(\tau),$$

$$\lim_{\tau_k\to +0} P^{(n)}_{\mathcal{K},t}(\tau) = I_{\mathcal{K}^a}(k) P^{(n)}_{\mathcal{K},t}(\tau) + \Big((W-I)\widetilde{v}(-t) + L\Big) P^{(n-1)}_{\mathcal{K}\setminus\{k\},t}(\tau). \tag{4.2}$$

Therefore, from (4.2) follows:

$$P^{(n)}_{\mathcal{K},t}(\tau)|^{\tau_k=-0}_{\tau_k=+0} = -\Big((W-I)\widetilde{v}(-t) + L\Big) P^{(n-1)}_{\mathcal{K}\setminus\{k\},t}(\tau)\, I_{\mathcal{K}}(k). \tag{4.3}$$

Let us find values of jumps of $\widetilde{\phi}_{n,t}(\tau)$ when τ_k changes the sign. Note that

$$\lim_{\tau_k\to -0} \widetilde{\phi}_{n,t}(\tau) = \widetilde{v}(-t)\widetilde{\phi}_{n-1,t}(\tau_1,\ldots,\tau_{k-1},\tau_{k+1},\ldots,\tau_n).$$

Taking into consideration equations (4.1) and (4.3) we obtain the amplitude and phase jumps for functions from the range of the resolvent of the unitary group U_t :

$$\lim_{\tau_k\to +0} \widetilde{\phi}_{n,t}(\tau) = W \lim_{\tau_k\to -0} \widetilde{\phi}_{n,t}(\tau) + L\,\widetilde{\phi}_{n-1,t}(\tau_1,\ldots,\tau_{k-1},\tau_{k+1},\ldots,\tau_n). \tag{4.4}$$

Denote by $\mathcal{D}_{W,L} = D\otimes\Gamma^S(\widetilde{W}^1_2(I\!R\setminus\{0\}))$ a vector subspace in $\mathbf{h}$ that satisfies the condition (4.4). By $A(\delta_\pm)$, $\Lambda(\delta_\pm)$, and $\widehat{N}$ we denote operators acting on symmetric Fock vectors as follows:

$$(\Phi,\Lambda(\delta_\pm)\Psi) = \lim_{\varepsilon\to\pm 0}\sum_1^\infty \frac{1}{n!}\sum_{k=1}^n \int_{\left(I\!R\setminus\{0\}\right)^{n-1}}$$

$$\times\prod_{m\neq k} d\tau_m (\widetilde{\Phi}_n,\widetilde{\Psi}_n)_{\mathcal{H}}(\tau_1,\ldots,\tau_{k-1},\epsilon,\tau_k,\ldots,\tau_{n-1}),$$

$$A(\delta_\pm\widetilde{\Psi})_n(\tau) = \lim_{\varepsilon\to\pm 0}\sum_{k=1}^n \widetilde{\Psi}_{n+1}(\tau_1,\ldots,\tau_{k-1},\epsilon,\tau_k,\ldots,\tau_n), \quad \widehat{N}\Psi_n(\omega) = n\Psi_n(\omega).$$

The boundary condition (4.4) in this notation looks as follows

$$(\widehat{N}+1)^{-1}(I\otimes A(\delta_+) - W\otimes A(\delta_-))\Psi = (L\otimes I)\,\Psi. \tag{4.5}$$

Let us prove that the operator

$$\widehat{\mathbf{H}} = iG\otimes I + I\otimes\widehat{\mathbf{E}} + iL^*W\otimes A(\delta_-), \quad \widehat{\mathbf{E}} = \mathcal{F}^*_{\tau\to\omega}\int_{I\!R\setminus\{0\}} d\tau a^+(\tau)a(\tau)\, i\partial_\tau \mathcal{F}_{\omega\to\tau}, \tag{4.6}$$

is symmetric in $\mathcal{D}_{W,L}$. Let $\Phi, \Psi \in \mathcal{D}_{W,L}$ and let B be a Hermitian operator so that $dom\, B \otimes I \supseteq \mathcal{D}_{W,L}$. Integration by parts implies an identity, where the difference of substitutions is expressed through operators $\Lambda(\delta_\pm)$:

$$(\Phi, B \otimes \widehat{\mathbf{E}}\, \Psi) - (B \otimes \widehat{\mathbf{E}}\, \Phi, \Psi) = i\Big(\Phi, B \otimes (\Lambda(\delta_-) - \Lambda(\delta_+))\Psi\Big) \qquad (4.7)$$

Using boundary condition (4.4) for functions $\widetilde{\Phi}_n$ and $\widetilde{\Psi}_n$ we find the difference between the substitutions (4.6):

$$i\Big(\Phi, B \otimes (\Lambda(\delta_+) - \Lambda(\delta_-))\Psi\Big) = i(\Phi, (W^* B W - B)\Lambda(\delta_-)\Psi)$$

$$+i(L\,\Phi, BL\,\Psi) + i(W\,A(\delta_-)\,\Phi, BL\,\Psi) + i(L\,\Phi, BW\,A(\delta_-)\,\Psi). \qquad (4.8)$$

In the particular case $B = I$, the equation (4.8) becomes simpler:

$$i\Big(\Phi, (\Lambda(\delta_+) - \Lambda(\delta_-))\Psi\Big) = i(\Phi, L^* L\Psi) - (iL^* W \otimes A(\delta_-)\Phi, \Psi) + (\Phi, iL^* W \otimes A(\delta_-)\Psi).$$

Now the identity $iG - iL^*L = (iG)^*$ and equation (4.7) prove that the operator $\widehat{\mathbf{H}}$ is symmetric on $\mathcal{D}_{W,L}$:

$$(\Phi, \widehat{\mathbf{H}}\Psi) = ((I \otimes \widehat{\mathbf{E}})\,\Phi, \Psi) + (\Phi, \{iG \otimes I + iL^* W \otimes A(\delta_-)\}\Psi)$$

$$-i(\Phi, I \otimes (\Lambda(\delta_+) - \Lambda(\delta_-))\Psi)$$

$$= ((I \otimes \widehat{\mathbf{E}})\,\Phi, \Psi) + (\Phi, (iG)^* \otimes I\Psi) + (iL^* W \otimes A(\delta_-)\Phi, \Psi)) = (\widehat{\mathbf{H}}\Phi, \Psi).$$

Let us find how the generator of the unitary group U_t acts on Fock vectors belonging to the range of the resolvent. Let $\Psi \in \mathbf{h}$, $\Phi = R_\mu h \otimes \psi(v)$. From the definition of the generator we have:

$$(\Psi, \widehat{\mathbf{H}}\,\Phi) = \lim_{s \to +0} \frac{1}{i}\frac{d}{ds}(\Psi, U_s \Phi) = \frac{1}{i}\int_0^\infty dt \sum_{n=0}^\infty \frac{1}{n!} \int_{\left(I\!R \setminus \{0\}\right)^n} d\tau$$

$$\times\Big(\widetilde{\psi}_n(\tau), \frac{d}{ds} e^{-(G+\mu)t - Gs - iL^* W \int_0^{t+s} \widetilde{v}(-\tau)\,d\tau}\, \widetilde{\phi}_{n,t+s}(\tau)\Big)_{\mathcal{H}}\Big|_{s=0}. \qquad (4.9)$$

Note that the functions $\widetilde{\phi}_{n,t}(\tau)$ depend on differences $\tau_k - t$. Hence,

$$\frac{d}{dt}\widetilde{\phi}_{n,t}(\tau) = -\sum_{k=1}^n \frac{\partial}{\partial \tau_k}\widetilde{\phi}_{n,t}(\tau) = i\mathcal{F}_{\omega \to \tau}\widehat{\mathbf{E}}\,\widetilde{\phi}_{n,t}(\tau). \qquad (4.10)$$

On the other hand, from the definition of the operator $A(\delta_-)$ we have

$$\mathcal{F}_{\omega\to\tau}(A(\delta_-)\phi_{n,t})(\tau) = n\widetilde{v}(-t)\,\phi_{n-1,t}(\tau). \tag{4.11}$$

Now from the (4.9) and form equations (4.10), (4.11) we obtain

$$(\Psi, \widehat{\mathbf{H}}\,\Phi)_{\mathbf{h}} = \int_0^\infty dt\,(\psi_0, iGe^{-(G+\mu)t-iL^*W\int_0^t \widetilde{v}(-\tau)\,d\tau}h)_{\mathcal{H}} + \int_0^\infty dt \sum_{n=1}^\infty \frac{1}{n!}\int_{(I\!R\setminus\{0\})^n} d\tau$$

$$\times\Big(\widetilde{\psi}_n(\tau), e^{-(G+\mu)t-iL^*W\int_0^t \widetilde{v}(-\tau)\,d\tau}(iG + iL^*W\widetilde{v}(-t) + i\sum_{k=1}^n \frac{\partial}{\partial\tau_k})\widetilde{\phi}_{n,t}(\tau)\Big)_{\mathcal{H}}$$

$$= \Big(\Psi, \Big\{iG + iL^*W\otimes A(\delta_-) + I\otimes\hat{\mathbf{E}}\Big\}\Phi\Big)_{\mathbf{h}},$$

that is the generator $\widehat{\mathbf{H}}$ of the group U_t coincides with the generator (4.5). Thus we have proved the theorem.

Theorem 4.1 *The generator* $\widehat{\mathbf{H}} = iG\otimes I + I\otimes\hat{\mathbf{E}} + iL^*W\otimes A(\delta_-)$ *of one-parameter unitary group* U_t *is symmetric in* $\mathcal{D}_{W,L}$.

We have proved that the operator $\widehat{\mathbf{H}}$ is symmetric without the assumption that operators L, G, and W commute. There is no conceptual difficulties to extend the construction of a symmetrical boundary value problem to a wider class of generators

$$\widehat{\mathbf{H}} = iG + I\otimes\hat{\mathbf{E}} + i\sum_{\ell,m} L_\ell^* W_{\ell,m}\otimes A_m(\delta_-) \tag{4.12}$$

with the boundary condition

$$(\widehat{N}+1)^{-1}(I\otimes A_\ell(\delta_+) - \sum_m W_{\ell,m}\otimes A_m(\delta_-))\Psi = (L_\ell\otimes I)\,\Psi, \tag{4.13}$$

where $W = \{W_{\ell,m}\}$ is $(M\times M)$ is a unitary matrices with coefficients from $\mathcal{B}(\mathcal{H})$, $\{A_\ell(g):\ g\in L_2(I\!R),\ 1\le\ell\le M\}$ are annihilation operators $\Gamma^S(L_2(I\!R^M))$ which commute for different indices ℓ, and $G = -iH_0 + \frac{1}{2}\sum L_\ell^* L_\ell$.

5 The Markow evolution equation

Let us consider how the Markow evolution equation can be derived from the boundary value problem for the Schrödinger equation

$$\frac{d}{dt}\Psi(t) = \Big(-G + iI\otimes\hat{\mathbf{E}} - \sum_{\ell,m} L_\ell^* W_{\ell,m}\otimes A_m(\delta_-)\Big)\Psi(t)$$

with the boundary condition (4.13). Let B be a Hermitian operator from $\mathcal{B}(\mathcal{H})$ and let $h,\, g\in D$. Consider an equation for mean values $(g, P_t(B)h)_{\mathcal{H}} = (U_t\, g\otimes\Psi(0)|B\otimes I|U_t\, h\otimes\Psi(0))_{\mathbf{h}}$. From (4.12) we have

$$\frac{d}{dt}(g, P_t(B)h)_{\mathcal{H}} = -((G+\sum_{\ell,m} L_\ell^* W_{\ell,m}\, A_m(\delta_-))U_t\, g\otimes\Psi(0)|B\otimes I|U_t\, h\otimes\Psi(0))$$

$$-(U_t\, g\otimes\Psi(0)|B\otimes I|(G+\sum_{\ell,m} L_\ell^* W_{\ell,m}\, A_m(\delta_-))U_t\, h\otimes\Psi(0))$$

$$+\Big(U_t\, g\otimes\Psi(0)|B\otimes(\Lambda(\delta_+)-\Lambda(\delta_-))|U_t\, h\otimes\Psi(0))\Big). \tag{4.14}$$

Now the equation (4.8) reads as

$$\Big(\Phi, B\otimes((\Lambda(\delta_+)-\Lambda(\delta_-))\Psi\Big) = \sum_{\ell,m}\Big(\Phi, (W_{\ell,m}^* B W_{l,m} - B)\Lambda_m(\delta_-)\Psi\Big)$$

$$+\sum_\ell (L_\ell\,\Phi, BL_\ell\,\Psi) + \sum_{\ell,m}\Big((W_{\ell,m}\, A_m(\delta_-)\,\Phi, BL_\ell\,\Psi) + (L_\ell\,\Phi, BW_{\ell,m}\, A_m(\delta_-)\,\Psi)\Big) \tag{4.15}$$

Since $A(\delta_-)\Psi(0)=0$, $\Lambda(\delta_-)\Psi(0)=0$, (4.14) and (4.15) imply:

$$\frac{d}{dt}(g, P_t(B)h)_{\mathcal{H}}\Big|_{t=0} = (g, \mathcal{L}(B)h) = -(Gg, Bh) - (g, BGh) + \sum_\ell (L_\ell g, BL_\ell g).$$

Hence, we obtain an infinitesimal map $\mathcal{L}(\cdot)$ of the Markov evolution equation in the standard Lindbladien form:

$$\frac{d}{dt}P_t(B) = \mathcal{L}(P_t(B)), \quad \mathcal{L}(B) = -G^*B - BG + \frac{1}{2}\sum_\ell L_\ell^* B L_\ell$$

for $G = -iH_0 + \frac{1}{2}\sum L_\ell^* L_\ell$.

6 Concluding remarks

Among the most important problems which stay unsolved in this paper we can mention

-a proof of *srs*-convergence $U_t^{(\alpha)} J_t^* \to u(0,t)$ for Hamiltonians with noncommuting coefficients H, R, and K;

-an extension of a theory to the case $\widehat{\mathbf{E}} = \int |\omega|^2 a^+(\omega)a(\omega)d\omega$ or $\widehat{\mathbf{E}} = \int(\omega|^2 + c^2)^{1/2} a^+(\omega)a(\omega)d\omega$, $\omega\in I\!R$;

-a study of conditions necessary and sufficient for the symmetric boundary problem described in §4 to be essentially self-adjoint;

-a generalization of quantum stochastic calculus to equations with nonadapted stochastic differentials extending the Ito multiplication table.

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